Singularity of the dyadic Green's function for heterogeneous dielectrics

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For heterogeneous media with piecewise constant complex permittivity on regular domains, we show that the dyadic Green's function has the same singular part as the corresponding free space dyadic Green's function on every domain of constant permittivity. We give two important applications of this property, namely the distorted-wave Born approximation for composite media and the calculation of the single photon decay rate in spontaneous emission.

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I. INTRODUCTION

In electromagnetism a key concept is the electric-field Green's function (actually a second rank tensor), also known as the dyadic Green's function. For a given real frequency ω, it represents (when multiplied by $\omega^2\mu_0$) the harmonic electric field radiated by three orthogonal point dipoles with time-dependence $\exp(-i\omega t)$. In a medium specified by the relative permittivity $\varepsilon(\boldsymbol{r},\omega)$, the Green's function is given by the outgoing solution $\mathsf{G}(\boldsymbol{r},\boldsymbol{r}',\omega)$ of the electric field equation with point current source term located at $\boldsymbol{r}'$,

$$\partial_{\boldsymbol{r}} \times \partial_{\boldsymbol{r}} \times \mathsf{G}(\boldsymbol{r},\boldsymbol{r}',\omega) - \frac{\omega^2}{c^2}\varepsilon(\boldsymbol{r},\omega)\mathsf{G}(\boldsymbol{r},\boldsymbol{r}',\omega) = \delta(\boldsymbol{r}-\boldsymbol{r}')\mathsf{I}, \tag{1}$$

where c is the light speed in vacuum and I is the unit 3×3 matrix. This outgoing solution $\mathsf{G}(\boldsymbol{r},\boldsymbol{r}',\omega)$ can be defined by $\mathsf{G}(\boldsymbol{r},\boldsymbol{r}',\omega+i0^+)$, the limit when $\eta\downarrow 0$ of $\mathsf{G}(\boldsymbol{r},\boldsymbol{r}',z)$, $z=\omega+i\eta$.

The determination of the Green's tensor for an arbitrary heterogeneous medium is quite complex since it amounts to solving a set of partial differential equations with nonconstant coefficients. It is known analytically or semianalytically in a few cases only, essentially the free-space, half-space, multilayer [1], and spherical [2,3] configurations. We recall the well-known expression of the vacuum Green's function $\mathsf{G}_0(\boldsymbol{r}-\boldsymbol{r}')$, away from the diagonal ($\boldsymbol{r}\neq\boldsymbol{r}'$), with $k_0=\omega/c$,

$$\mathsf{G}_0(\boldsymbol{r}) = \frac{e^{ik_0 r}}{4\pi r}\left[\left(1+\frac{i}{k_0 r}-\frac{1}{k_0^2 r^2}\right)\mathsf{I} - \left(1+\frac{3i}{k_0 r}-\frac{1}{k_0^2 r^2}\right)\hat{\boldsymbol{r}}\otimes\hat{\boldsymbol{r}}\right]. \tag{2}$$

We use the following notations for all three-components vector $\boldsymbol{a}$: $\hat{\boldsymbol{a}}=\boldsymbol{a}/a$, $a^2=\boldsymbol{a}\cdot\boldsymbol{a}$, and $\hat{\boldsymbol{a}}\otimes\hat{\boldsymbol{a}}$ is the tensor product of $\hat{\boldsymbol{a}}$ with itself. In view of the nonintegrable $1/r^3$ singularity it is customary to decompose G into a singular and principal value part,

$$\mathsf{G}(\boldsymbol{r},\boldsymbol{r}') = \frac{c^2}{\omega^2}\mathsf{L}\,\delta(\boldsymbol{r}-\boldsymbol{r}') + \mathsf{G}^{pv}(\boldsymbol{r},\boldsymbol{r}'). \tag{3}$$

Here G^{pv} is the principal value for an exclusion domain V_η of a given shape (sphere, cube, etc.), defined by

$$\int d\boldsymbol{r}\mathsf{G}^{pv}(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{f}(\boldsymbol{r}) = \lim_{V_\eta\downarrow 0}\int_{\boldsymbol{r}\notin V_\eta} d\boldsymbol{r}\mathsf{G}^{pv}(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{f}(\boldsymbol{r}) \tag{4}$$

for all vector-valued functions $\boldsymbol{f}$, and L is a constant tensor depending on the shape of the exclusion volume (see [4] for a complete discussion). The decomposition, Eq. (3), is in general unknown for nonconstant permittivity.

The behavior of the Green's tensor around its diagonal turns out to have physical implications, of which we give two examples in the next section: the distorted-wave Born approximation (DWBA) in scattering calculations (case A) and the single photon radiative decay rate (case B). Therefore it is important to derive a correct decomposition of the type (3). If the permittivity remains constant over some domain, it is often claimed that the Green's tensor in this domain has the same behavior around its diagonal as the free space Green's tensor with corresponding permittivity (e.g., Ref. [5]). The usual heuristic argument is that the Green's tensor can be decomposed into a direct and indirect field: the former expresses the free propagation from the source to the observation point in the homogeneous domain and the latter, nonsingular contribution, accounts for the reflections on the interface of the domain of constant permittivity.

The aim of the present work is to establish this assertion in a rigorous and precise way. We will prove the following result, which we refer to as the "Rule of Piecewise Constant Permittivity" (RPCP):

Consider anheterogeneous dielectric medium described by a piecewise constant relative permittivity $\varepsilon(\boldsymbol{r})$, with nonnegative imaginary part $[\mathrm{Im}\,\varepsilon(\boldsymbol{r})\geqslant 0]$ and constant value $\varepsilon_{\mathcal{D}}$ on a domain $\mathcal{D}$. Let G be the corresponding Green's tensor and let $\mathsf{G}_{\mathcal{D}}$ be the free-space Green's tensor associated to the constant permittivity $\varepsilon_{\mathcal{D}}$. Then, for fixed $\boldsymbol{r}'$, the difference of

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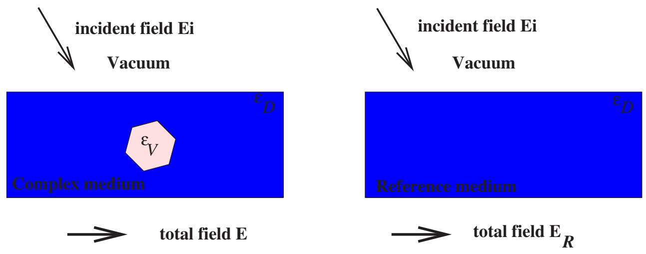

FIG. 1. (Color online) Schematic representation of the complex (left) and reference (right) scattering problem. The reference medium is heterogeneous but has well-characterized electric field solution and Green's function. This is the case for simple geometries (half-plane, multilayer, sphere, etc.).

the kernels $\mathsf{G}(\cdot,\boldsymbol{r}')-\mathsf{G}_{\mathcal{D}}(\cdot,\boldsymbol{r}')$ *is nonsingular in the domain* $\mathcal{D}$.

We will actually prove that the tensor-valued function $\mathsf{G}-\mathsf{G}_{\mathcal{D}}$ is infinitely differentiable. A consequence of this result is that the tensor carrying the singular part in the decomposition (3) is that of an infinite free space with the local dielectric constant.

II. CONSEQUENCES OF THE RULE OF LOCALLY CONSTANT PERMITTIVITY

A. Problem A: Distorted-wave Born approximation

A powerful way to handle complex heterogeneous media is to consider them as perturbation of well-characterized reference problems. To be more specific, suppose there is a small heterogeneity region of arbitrary shape $\mathcal{V}$ and relative permittivity $\varepsilon_{\mathcal{V}}$ embedded in a homogeneous host medium $\mathcal{D}$ with relative permittivity $\varepsilon_{\mathcal{D}}$, surrounded by vacuum (Fig. 1). Denote by $\varepsilon(\boldsymbol{r})$ and $\varepsilon_{\mathcal{R}}(\boldsymbol{r})$ the overall relative permittivity of the composite and reference medium, respectively,

$$\varepsilon(\boldsymbol{r})=\begin{cases}\varepsilon_{\mathcal{V}}, & \boldsymbol{r}\in\mathcal{V},\\ \varepsilon_{\mathcal{D}}, & \boldsymbol{r}\in\mathcal{D},\quad \boldsymbol{r}\notin\mathcal{V},\\ 1, & \text{all other cases},\end{cases}\tag{5}$$

and

$$\varepsilon_{\mathcal{R}}(\boldsymbol{r})=\begin{cases}\varepsilon_{\mathcal{D}}, & \boldsymbol{r}\in\mathcal{D},\\ 1, & \text{all other cases}.\end{cases}\tag{6}$$

Now assume the composite medium is illuminated by some incident beam $\boldsymbol{E}_i$, and let $\boldsymbol{E}$ be the total resulting electric field. We denote $\boldsymbol{E}_{\mathcal{R}}$ the field produced by the same incident beam for the reference medium. Then we have the well-known electric field equation

$$\boldsymbol{E}(\boldsymbol{r}')=\boldsymbol{E}_{\mathcal{R}}(\boldsymbol{r}')+\frac{\omega^2}{c^2}\int d\boldsymbol{r}\mathsf{G}_{\mathcal{R}}(\boldsymbol{r},\boldsymbol{r}')\chi(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r}),\tag{7}$$

where $\chi=\varepsilon-\varepsilon_{\mathcal{R}}$ is the relative dielectric contrast and $\mathsf{G}_{\mathcal{R}}$ the dyadic Green's function of the reference problem, that is the outgoing solution of (1) with ε replaced by $\varepsilon_{\mathcal{R}}$.

Using the decomposition (3) we rewrite

$$[\mathsf{I}-\mathsf{L}(\boldsymbol{r}')\chi(\boldsymbol{r}')]\boldsymbol{E}(\boldsymbol{r}')=\boldsymbol{E}_{\mathcal{R}}(\boldsymbol{r}')+\frac{\omega^2}{c^2}\int d\boldsymbol{r}\mathsf{G}^{pv}_{\mathcal{R}}(\boldsymbol{r},\boldsymbol{r}')\chi(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r}),\tag{8}$$

where the principal value integral is calculated in the limit of small exclusion volume $\mathcal{V}_\eta$. Now if we choose as exclusion volume the heterogeneity $\mathcal{V}$ itself (assuming the latter is small enough to achieve the principal value), we exclude precisely the support of the contrast function χ and force the corresponding integral to zero. We thus have the following approximation for the field at any point inside the source region $\mathcal{V}$:

$$\boldsymbol{E}(\boldsymbol{r}')\simeq[\mathsf{I}-\mathsf{L}(\boldsymbol{r}')\chi(\boldsymbol{r}')]^{-1}\boldsymbol{E}_{\mathcal{R}}(\boldsymbol{r}').\tag{9}$$

This is often referred to as the distorted-wave Born approximation (DWBA), since it amounts to replacing the unknown field in the heterogeneous region by the "distorted" incident field, which is the reference field. The latter is in general well characterized. For example, it is given by the Mie solution for a reference medium consisting of a homogeneous dielectric sphere embedded in vacuum, or is known perturbatively for a homogeneous domain with rough boundaries (e.g., Ref. [6]). This raises a question about the *a priori* knowledge of the deltalike tensor L in the singular part of the reference Green's function. Now the answer is a consequence of the rule of piecewise constant permittivity: since ε assumes the constant value $\varepsilon_{\mathcal{D}}$ in the domain $\mathcal{D}$, the singularity tensor L in that region is that of the free-space Green's function in a homogeneous medium with permittivity $\varepsilon_{\mathcal{D}}$. In particular, for a spherical or cubical exclusion domain $\mathsf{L}(\boldsymbol{r}')=-1/(3\varepsilon_{\mathcal{D}})\mathsf{I}$. If $\mathcal{D}$ is the reference problem for a composite medium with spherical or cubical inclusion $\mathcal{V}$, we thus have for the DWBA in the source region

$$\boldsymbol{E}(\boldsymbol{r}')\simeq\frac{3\varepsilon_{\mathcal{D}}}{2\varepsilon_{\mathcal{D}}+\varepsilon_{\mathcal{V}}}\boldsymbol{E}_{\mathcal{R}}(\boldsymbol{r}').\tag{10}$$

This gives, for example, a good approximation of the internal field in a small sphere embedded in a larger homogeneous object of arbitrary shape.

B. Problem B: Calculation of the single photon decay rate

The evaluation of the Green's function at the source position is of fundamental importance in quantum electrodynamics as it determines the radiative properties of an excited atom in a given host medium. The imaginary part of the Green's function gives the local density of states (LDOS), which is proportional to the single photon decay rate. Its real part is proportional to the level shift of energy (Lamb shift [7]). We recall the well-known formula derived from the Fermi golden rule for the spontaneous decay rate of an excited atom located at position $\boldsymbol{r}'$ inside a linear dielectric material of Green's function G,

$$\Gamma=\frac{2\omega^2}{\hbar\varepsilon_0 c^2}\,\mathrm{Im}[\boldsymbol{p}_0\cdot\mathsf{G}(\boldsymbol{r}',\boldsymbol{r}')\cdot\boldsymbol{p}_0],\tag{11}$$

where $\boldsymbol{p}_0$ is a random unit dipolar moment. The calculation can be carried out for a homogeneous dielectric nonabsorptive material ($\varepsilon_{\mathcal{D}}$):

$$\mathrm{Im}[\boldsymbol{p}_0 \cdot \mathsf{G}(\boldsymbol{r}',\boldsymbol{r}') \cdot \boldsymbol{p}_0] = \frac{\omega}{6\pi c}\sqrt{\varepsilon_{\mathcal{D}}}. \tag{12}$$

However, diverging expressions are obtained for absorptive materials. Recently, the definition of the LDOS has been extended to situations where absorptive materials are present [3,8–15]: it is given by the imaginary part of the "transverse Green's function" $\mathsf{G}^{\perp}$ (see Sec. III for the definition). Hence it is crucial to show, first, that this latter quantity is actually finite and, second, how to calculate it from the total Green's function in heterogeneous media. We will now consider these two questions in the light of the rule of piecewise constant permittivity (RPCP). Denote as before $\varepsilon(\boldsymbol{r})$ and $\varepsilon_{\mathcal{D}}$ the relative permittivities of the heterogeneous and homogeneous media, respectively, with $\varepsilon=\varepsilon_{\mathcal{D}}$ on a domain $\mathcal{D}$, and let again G, $\mathsf{G}_{\mathcal{D}}$ be the associated Green's functions. The difference $\mathsf{G}(\cdot,\boldsymbol{r}')-\mathsf{G}_{\mathcal{D}}(\cdot,\boldsymbol{r}')$ is infinitely differentiable around $\boldsymbol{r}'$, as well as its transverse part. The imaginary part of $\mathsf{G}^{\perp}_{\mathcal{D}}(\boldsymbol{r}',\boldsymbol{r}')$ can be determined analytically [9]. Inverting the Helmholtz operator in the second Eq. (19) (see Sec. III) we obtain

$$\begin{aligned}\mathsf{G}^{\perp}_{\mathcal{D}}(\boldsymbol{r}',\boldsymbol{r}') &= \frac{1}{(2\pi)^3}\int d\boldsymbol{p}\left[p^2-\frac{\omega^2}{c^2}\varepsilon_{\mathcal{D}}\right]^{-1}\{\mathsf{I}-\hat{\boldsymbol{p}}\otimes\hat{\boldsymbol{p}}\}\\ &= \frac{1}{(2\pi)^3}\frac{2}{3}\int d\boldsymbol{p}\left[p^2-\frac{\omega^2}{c^2}\varepsilon_{\mathcal{D}}\right]^{-1}\mathsf{I}\\ &= \frac{1}{(2\pi)^3}\frac{4\pi}{3}\int_{-\infty}^{\infty} dp\frac{p^2}{p^2-\frac{\omega^2}{c^2}\varepsilon_{\mathcal{D}}}\mathsf{I},\end{aligned} \tag{13}$$

leading to

$$\begin{aligned}\mathrm{Im}\,\mathsf{G}^{\perp}_{\mathcal{D}}(\boldsymbol{r}',\boldsymbol{r}') &= \frac{1}{(2\pi)^3}\frac{4\pi}{3}\int_{-\infty}^{\infty} dp\frac{p^2\frac{\omega^2}{c^2}\,\mathrm{Im}\,\varepsilon_{\mathcal{D}}}{\left[p^2-\frac{\omega^2}{c^2}\varepsilon_{\mathcal{D}}\right]^2}\mathsf{I}\\ &= \frac{\omega}{6\pi c}\,\mathrm{Re}\,\sqrt{\varepsilon_{\mathcal{D}}}\mathsf{I},\end{aligned} \tag{14}$$

which is finite in both absorptive and nonabsorptive cases. We can therefore conclude that the imaginary part of $\mathsf{G}^{\perp}(\boldsymbol{r}',\boldsymbol{r}')$ as well as the LDOS always exist. Furthermore, from Eqs. (19) and (23) (see Sec. III) we easily obtain the longitudinal and transverse Green's functions,

$$\mathsf{G}^{\parallel}(\boldsymbol{r},\boldsymbol{r}') = \frac{\varepsilon_{\mathcal{D}}}{\varepsilon(\boldsymbol{r})}\mathsf{G}^{\parallel}_{\mathcal{D}}(\boldsymbol{r},\boldsymbol{r}') = -\left[\frac{\omega^2}{c^2}\varepsilon(\boldsymbol{r})\right]^{-1}\mathsf{P}^{\parallel}\delta(\boldsymbol{r}-\boldsymbol{r}'),$$

$$\mathsf{G}^{\perp}(\boldsymbol{r},\boldsymbol{r}') = \mathsf{G}(\boldsymbol{r},\boldsymbol{r}') + \left[\frac{\omega^2}{c^2}\varepsilon(\boldsymbol{r})\right]^{-1}\mathsf{P}^{\parallel}\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{15}$$

Hence the knowledge of $\mathsf{G}^{\perp}$ is equivalent to that of G. In particular, if there is no absorption at $\boldsymbol{r}'$ [$\mathrm{Im}\,\varepsilon(\boldsymbol{r}')=0$], one has $\mathrm{Im}\,\mathsf{G}^{\perp}=\mathrm{Im}\,\mathsf{G}$ since then the longitudinal part is real.

III. DERIVATION OF THE MAIN RESULT

To highlight the main ideas of the derivation, we will provide a short heuristic proof. A completely rigorous one can be found in the Appendix. The method relies essentially on the decomposition of the electric field into transverse and longitudinal parts. This is commonly achieved by writing a vector field $\boldsymbol{f}(\boldsymbol{r})$ as the sum of the gradient of a scalar potential and the rotation of a vector potential. But not every square integrable $\boldsymbol{f}(\boldsymbol{r})$ is differentiable so we give a general definition in terms of its Fourier transform,

$$\boldsymbol{f}(\boldsymbol{p}) = \frac{1}{(2\pi)^3}\int d\boldsymbol{r}\,\exp[-i\boldsymbol{p}\cdot\boldsymbol{r}]\boldsymbol{f}(\boldsymbol{r}). \tag{16}$$

The longitudinal and transverse parts $\boldsymbol{f}^{\parallel}(\boldsymbol{r})$ and $\boldsymbol{f}^{\perp}(\boldsymbol{r})$ of $\boldsymbol{f}(\boldsymbol{r})$ are defined by

$$\boldsymbol{f}^{\parallel}(\boldsymbol{r}) = (\mathsf{P}^{\parallel}\boldsymbol{f})(\boldsymbol{r}) = \int d\boldsymbol{p}\,\exp[i\boldsymbol{p}\cdot\boldsymbol{r}]\hat{\boldsymbol{p}}\otimes\hat{\boldsymbol{p}}\boldsymbol{f}(\boldsymbol{p}),$$

$$\boldsymbol{f}^{\perp}(\boldsymbol{r}) = (\mathsf{P}^{\perp}\boldsymbol{f})(\boldsymbol{r}) = \boldsymbol{f}(\boldsymbol{r}) - \boldsymbol{f}^{\parallel}(\boldsymbol{r}). \tag{17}$$

In coordinate space $\boldsymbol{p}=-i\partial_{\boldsymbol{r}}$, we have $\partial_{\boldsymbol{r}}\times\partial_{\boldsymbol{r}}=p^2\mathsf{I}-\boldsymbol{p}\otimes\mathbf{p}=p^2\mathsf{P}^{\perp}$, and thus

$$\begin{aligned}(\mathsf{H}_0\boldsymbol{f})(\boldsymbol{r}) &= (\partial_{\boldsymbol{r}}\times\partial_{\boldsymbol{r}}\times\boldsymbol{f})(\boldsymbol{r})\\ &= (p^2\mathsf{P}^{\perp}\boldsymbol{f})(\boldsymbol{r}) = (\boldsymbol{p}^2\boldsymbol{f}^{\perp})(\boldsymbol{r}) = -\Delta_{\boldsymbol{r}}\boldsymbol{f}^{\perp}(\boldsymbol{r}).\end{aligned} \tag{18}$$

Hence Eq. (1) for the Green's tensors can be rewritten as

$$\Delta_{\boldsymbol{r}}\mathsf{G}^{\perp}(\boldsymbol{r},\boldsymbol{r}') + \frac{\omega^2}{c^2}\varepsilon(\boldsymbol{r})\mathsf{G}(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}')\mathsf{I},$$

$$\Delta_{\boldsymbol{r}}\mathsf{G}^{\perp}_{\mathcal{D}}(\boldsymbol{r},\boldsymbol{r}') + \frac{\omega^2}{c^2}\varepsilon_{\mathcal{D}}\mathsf{G}_{\mathcal{D}}(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}')\mathsf{I}, \tag{19}$$

where $\mathsf{G}^{\perp}(\boldsymbol{r},\boldsymbol{r}')=(\mathsf{P}^{\perp}\mathsf{G})(\boldsymbol{r},\boldsymbol{r}')$. Subtracting the previous equations from one another and introducing the difference kernel:

$$\mathsf{U}(\boldsymbol{r},\boldsymbol{r}') = \mathsf{G}(\boldsymbol{r},\boldsymbol{r}') - \mathsf{G}_{\mathcal{D}}(\boldsymbol{r},\boldsymbol{r}'), \tag{20}$$

leads to

$$\Delta_{\boldsymbol{r}}\mathsf{U}^{\perp}(\boldsymbol{r},\boldsymbol{r}') + \varepsilon(\boldsymbol{r})\frac{\omega^2}{c^2}\mathsf{U}(\boldsymbol{r},\boldsymbol{r}') = \frac{\omega^2}{c^2}[\varepsilon_D-\varepsilon(\boldsymbol{r})]\mathsf{G}_D(\boldsymbol{r},\boldsymbol{r}'). \tag{21}$$

Next we separate the transverse and longitudinal components of this last equation. This is possible, provided for all square integrable $\boldsymbol{f}(\boldsymbol{r})$,

$$\varepsilon(\boldsymbol{r})\boldsymbol{f}^{\perp}(\boldsymbol{r}) = [\varepsilon(\boldsymbol{r})\boldsymbol{f}(\boldsymbol{r})]^{\perp}, \quad \varepsilon(\boldsymbol{r})\boldsymbol{f}^{\parallel}(\boldsymbol{r}) = [\varepsilon(\boldsymbol{r})\boldsymbol{f}(\boldsymbol{r})]^{\parallel}, \tag{22}$$

in other words, if the permittivity function commutes with the transverse and longitudinal projection. This is actually the case if ε is piecewise constant ([16], see the lemma in Appendix) and if the fields involved are square integrable. In that case we obtain for U the couple of equations,

$$\left[\Delta_{\boldsymbol{r}}+\varepsilon(\boldsymbol{r})\frac{\omega^2}{c^2}\right]\mathsf{U}^{\perp}(\boldsymbol{r},\boldsymbol{r}') = \frac{\omega^2}{c^2}[\varepsilon_{\mathcal{D}}-\varepsilon(\boldsymbol{r})]\mathsf{G}^{\perp}_{\mathcal{D}}(\boldsymbol{r},\boldsymbol{r}'),$$

$$\varepsilon(\boldsymbol{r})\mathsf{U}^{\parallel}(\boldsymbol{r},\boldsymbol{r}')=[\varepsilon_{\mathcal{D}}-\varepsilon(\boldsymbol{r})]\mathsf{G}^{\parallel}_{\mathcal{D}}(\boldsymbol{r},\boldsymbol{r}'). \quad (23)$$

It follows that the longitudinal part $\mathsf{U}^{\parallel}$ vanishes on $\mathcal{D}$. On the other hand, $\mathsf{U}^{\perp}(\boldsymbol{r},\boldsymbol{r}')$ is a solution of the Helmholtz equation in $\mathcal{D}$,

$$\left[\Delta_{\boldsymbol{r}}+\varepsilon_{\mathcal{D}}\frac{\omega^2}{c^2}\right]\mathsf{U}^{\perp}(\boldsymbol{r},\boldsymbol{r}')=0, \quad (24)$$

and $\mathsf{U}^{\perp}(\boldsymbol{r},\boldsymbol{r}')$ is therefore regular (actually infinitely differentiable, see e.g., Ref. [17], Theorem IX.26) in this domain.

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APPENDIX: A MATHEMATICAL PROOF

We now provide a rigorous proof of the "Rule of Piecewise Constant Permittivity" in the mathematical framework of linear operator theory. Let $\mathcal{H}=L^2(\mathbb{R}^3,d\boldsymbol{r};\mathbb{C}^3)$ be the Hilbert space of square-integrable functions over $\mathbb{R}^3$ with values in $\mathbb{C}^3$. On $\mathcal{H}$ we define the inverse operators

$$\mathsf{R}_0(z)=[\mathsf{H}_0-\varepsilon_{\mathcal{D}}z^2]^{-1},$$

$$\mathsf{R}_\chi(z)=[\mathsf{H}_0-(1+\chi)\varepsilon_{\mathcal{D}}z^2]^{-1}. \quad (A1)$$

Note that the kernels of these operators correspond to the Green's tensors $\mathsf{G}_{\mathcal{D}}$ and G when $z^2=\omega^2/c^2$ and $(1+\chi)=\varepsilon/\varepsilon_{\mathcal{D}}$. We will show that the difference of these kernels is nonsingular for locally constant permittivity.

These two inverse operators are bounded for $\operatorname{Im} z>0$ (for instance $z=\omega/c+i\eta$, $\eta>0$) and $\operatorname{Im}\varepsilon\geqslant 0$. Indeed, the spectrum of the Helmholtz operators $[H_0-\varepsilon_{\mathcal{D}}z^2]$ and $[H_0-(1+\chi)\varepsilon_{\mathcal{D}}z^2]$ is included in the lower part of the complex plane [18]: the complex resonances of the structure have negative imaginary part and therefore

$$\|\mathsf{R}_0(z)\|\leqslant\eta^{-1}=[\operatorname{Im} z]^{-1},\quad \|\mathsf{R}_\chi(z)\|\leqslant\eta^{-1}. \quad (A2)$$

From a physical point of view, this property can be deduced from the conservation ($\operatorname{Im}\varepsilon=0$ and real resonances) or the attenuation ($\operatorname{Im}\varepsilon\geqslant 0$ and complex resonances in the lower half-plane) of the electromagnetic field energy with time dependence $\exp[-i\omega t]$.

Since H_0 commutes with both $\mathsf{P}^{\perp}$ and $\mathsf{P}^{\parallel}$, the same holds for its resolvent,

$$\mathsf{P}^{\parallel}\mathsf{R}_0(z)=\mathsf{R}_0(z)\mathsf{P}^{\parallel}=-[\varepsilon_{\mathcal{D}}z^2]^{-1}\mathsf{P}^{\parallel},$$

$$\mathsf{P}^{\perp}\mathsf{R}_0(z)=\mathsf{R}_0(z)\mathsf{P}^{\perp}=-[\Delta+\varepsilon_{\mathcal{D}}z^2]^{-1}\mathsf{P}^{\perp}. \quad (A3)$$

It is *a priori* not obvious whether $R_\chi(z)$ shares these commutation properties with the free resolvent. This turns out to be the case under certain conditions on χ. We use the following lemma established and proven in [16].

Lemma [16]. Let $\mathcal{A}\subset\mathbb{R}^3$ have "regular boundaries" $\partial\mathcal{A}$ and let $\mathsf{I}_{\mathcal{A}}$ be its characteristic function, $\mathsf{I}_{\mathcal{A}}(\boldsymbol{r})=1$ for $\boldsymbol{r}\in\mathcal{A}$ and 0 otherwise. Let $\mathsf{P}_{\mathcal{A}}$ the projector associated with $\mathsf{I}_{\mathcal{A}}(\boldsymbol{r})$. Then $\mathsf{P}_{\mathcal{A}}$ and $\mathsf{P}^{\parallel}$ (hence $\mathsf{P}_{\mathcal{A}}$ and $\mathsf{P}^{\perp}$) commute. (With "regular boundaries" we mean that $\partial\mathcal{A}$ is Lebesgue measurable with zero Lebesgue measure, like the boundaries of spheres, cubes etc., and planar interfaces.)

A direct consequence of this lemma is that any piecewise constant function χ on a regular domain commutes with the longitudinal projector. Hence Eq. (A3) holds also for $\mathsf{R}_\chi(z)$, with $\varepsilon_{\mathcal{D}}$ replaced by $(1+\chi)\varepsilon_{\mathcal{D}}=\varepsilon$.

From the first resolvent identity, we have the difference

$$\mathsf{W}(z)=\mathsf{R}_\chi(z)-\mathsf{R}_0(z)=\mathsf{R}_\chi(z)\chi\varepsilon_{\mathcal{D}}z^2\mathsf{R}_0(z). \quad (A4)$$

Since $\mathsf{R}_\chi(z)$ and $\mathsf{R}_0(z)$ commute with the projectors $\mathsf{P}^{\perp}$ and $\mathsf{P}^{\parallel}$, this yields the decomposition

$$\mathsf{W}(z)=\mathsf{W}^{\parallel}(z)+\mathsf{W}^{\perp}(z),$$

$$\mathsf{W}^{\parallel}(z)=\mathsf{P}^{\parallel}\mathsf{W}(z)=\mathsf{W}(z)\mathsf{P}^{\parallel},$$

$$\mathsf{W}^{\perp}(z)=\mathsf{P}^{\perp}\mathsf{W}(z)=\mathsf{W}(z)\mathsf{P}^{\perp}. \quad (A5)$$

The expression of the longitudinal part is easily obtained

$$\mathsf{W}^{\parallel}(z)=-z^{-2}\frac{\chi}{1+\chi}\mathsf{P}^{\parallel}, \quad (A6)$$

which can be written as a deltalike kernel,

$$[\mathsf{W}^{\parallel}(z)\boldsymbol{f}](\boldsymbol{r}')=\int d\boldsymbol{r}\,\mathsf{U}^{\parallel}(\boldsymbol{r},\boldsymbol{r}',z)\boldsymbol{f}(\boldsymbol{r}), \quad (A7)$$

with

$$\mathsf{U}^{\parallel}(\boldsymbol{r},\boldsymbol{r}',z)=-z^{-2}\frac{\chi(\boldsymbol{r})}{1+\chi(\boldsymbol{r})}\delta^{\parallel}(\boldsymbol{r}-\boldsymbol{r}'). \quad (A8)$$

Note that $\mathsf{U}^{\parallel}(\boldsymbol{r},\boldsymbol{r}',z)$ vanishes for $\boldsymbol{r}\in\mathcal{D}$ since the support of χ is the complement of $\mathcal{D}$ in $\mathbb{R}^3$.

Next we turn to the transverse part $\mathsf{W}^{\perp}(z)$. First it is important to note that it is the product of two bounded Hilbert-Schmidt operators ([19], Sec. VI.6). Indeed we have

$$\|\mathsf{P}^{\perp}\mathsf{R}_\chi(z)\|=\|[\Delta+(1+\chi)\varepsilon_{\mathcal{D}}z^2]^{-1}\mathsf{P}^{\perp}\|\leqslant$$
$$\|[\Delta+(1+\chi)\varepsilon_{\mathcal{D}}z^2]^{-1}\|\leqslant\alpha$$
$$\|[-\Delta+\eta]^{-1}\|,\quad \alpha\geqslant 1, \quad (A9)$$

since, $\mathsf{P}^{\perp}\mathsf{R}_\chi(z)$ is bounded by η^{-1} [as well as $\mathsf{R}_\chi(z)$ (A2)] and, in addition, has the same behavior as Δ for high momentum. Thus,

$$\mathsf{W}^{\perp}(z)=\mathsf{P}^{\perp}\mathsf{R}_\chi(z)\chi\varepsilon_{\mathcal{D}}z^2\mathsf{P}^{\perp}\mathsf{R}_0(z) \quad (A10)$$

is bounded by $\beta[-\Delta+\eta]^{-2}$ (with β a positive number) and is therefore Hilbert-Schmidt. It follows (Theorem VI.23 [19]) that it can be represented by a square-integrable kernel $\mathsf{U}^{\perp}(\boldsymbol{r},\boldsymbol{r}',z)$:

$$[\mathsf{W}^{\perp}(z)\boldsymbol{f}](\boldsymbol{r}')=\int d\boldsymbol{r}\,\mathsf{U}^{\perp}(\boldsymbol{r},\boldsymbol{r}',z)\boldsymbol{f}(\boldsymbol{r}). \quad (A11)$$

The expression of the kernel is generally given with "bra and ket" notations,

$$\mathsf{U}^{\perp}(\boldsymbol{r},\boldsymbol{r}',z) = \frac{1}{(2\pi)^3}\int d\boldsymbol{p}\,\exp[i\boldsymbol{p}\cdot\boldsymbol{r}']\{\mathsf{W}^{\perp}(z)\exp[-i\boldsymbol{p}\cdot\boldsymbol{r}]\} = \langle\boldsymbol{r}'|\mathsf{W}^{\perp}(z)|\boldsymbol{r}\rangle, \tag{A12}$$

where $|\boldsymbol{r}\rangle$ is the multiplication by the function $\boldsymbol{p}\mapsto(2\pi)^{-3/2}\exp[-i\boldsymbol{p}\cdot\boldsymbol{r}]$ and $\langle\boldsymbol{r}'|$ is the linear operator $\boldsymbol{f}(\boldsymbol{p})\mapsto(2\pi)^{-3/2}\int d\boldsymbol{p}\,\exp[i\boldsymbol{p}\cdot\boldsymbol{r}']\boldsymbol{f}(\boldsymbol{p})$. Let $\mathcal{A}\subset\mathbb{R}^3$ have "regular boundaries" and satisfy the hypothesis of the lemma. From (A10) and the commutation property we have

$$\mathsf{P}_{\mathcal{A}}[\Delta+\{1+\chi\}\varepsilon_{\mathcal{D}}z^2]\mathsf{W}^{\perp}(z) = \mathsf{I}_{\mathcal{A}}\chi\varepsilon_{\mathcal{D}}z^2[\Delta+\varepsilon_{\mathcal{D}}z^2]^{-1}. \tag{A13}$$

Again, this operator is bounded to the left by $\gamma[-\Delta+\eta]^{-1}$ (with γ a positive number) and has a well-defined kernel. Consequently, the function

$$\mathsf{P}_{\mathcal{A}}[\Delta+\{1+\chi\}\varepsilon_{\mathcal{D}}z^2]\mathsf{W}^{\perp}(z)\exp[-i\boldsymbol{p}\cdot\boldsymbol{r}]$$

is square integrable so we can commute the integral in (A12) and the operator $\mathsf{P}_{\mathcal{A}}[\Delta+\{1+\chi\}\varepsilon_{\mathcal{D}}z^2]$:

$$\mathsf{P}_{\mathcal{A}}[\Delta+\{1+\chi\}\varepsilon_{\mathcal{D}}z^2]\mathsf{U}^{\perp}(\boldsymbol{r},\boldsymbol{r}',z) = \langle\boldsymbol{r}'|\mathsf{P}_{\mathcal{A}}[\Delta+\{1+\chi\}\varepsilon_{\mathcal{D}}z^2]\mathsf{W}^{\perp}(z)|\boldsymbol{r}\rangle = \varepsilon_{\mathcal{D}}z^2\langle\boldsymbol{r}'|\mathsf{I}_{\mathcal{A}}\chi[\Delta+\varepsilon_{\mathcal{D}}z^2]^{-1}|\boldsymbol{r}\rangle. \tag{A14}$$

With $\mathcal{A}=\mathcal{D}$ the complement of the support of χ, $\mathsf{I}_{\mathcal{A}}\chi=\mathsf{I}_{\mathcal{D}}\chi=0$ and, for all $\boldsymbol{r}\in\mathbb{R}^3$ and fixed $\boldsymbol{r}'\in\mathcal{D}$,

$$\mathsf{I}_{\mathcal{D}}[\Delta+\varepsilon_{\mathcal{D}}z^2]\mathsf{U}^{\perp}(\boldsymbol{r},\boldsymbol{r}',z)=0. \tag{A15}$$

Finally, for real values of the frequency, it is possible to take the weak limit $\eta=\mathrm{Im}\,z\downarrow 0$ of the above equation leading to

$$\mathsf{I}_{\mathcal{D}}[\Delta+\varepsilon_{\mathcal{D}}\omega^2/c^2]\mathsf{U}^{\perp}(\boldsymbol{r},\boldsymbol{r}',\omega/c)=0 \tag{A16}$$

in the sense of distributions. It follows (see e.g., Ref. [17], Theorem IX.26) that $\mathsf{U}^{\perp}(\boldsymbol{r},\boldsymbol{r}',z)$ is a C^{∞}-function of $\boldsymbol{r}$ in the domain $\mathcal{D}$ for $\mathrm{Im}\,z\geqslant 0$. This establishes the smoothness with respect to the variable $\boldsymbol{r}$ of $\mathsf{U}(\boldsymbol{r},\boldsymbol{r}',z)=\mathsf{G}_{\mathcal{D}}(\boldsymbol{r},\boldsymbol{r}',z)-\mathsf{G}(\boldsymbol{r},\boldsymbol{r}',z)$.

[1] W. Chew, *Waves and Fields in Inhomogeneous Media*, IEEE Press Series on Electromagnetic Wave Theory (IEEE, New York, 1999).
[2] C. Tai, *Dyadic Green's Functions in Electromagnetic Theory* (Intext, San Francisco, 1971).
[3] A. Tip, Phys. Rev. A **56**, 5022 (1997).
[4] D. Yaghjian, Proc. IEEE **68**, 248 (1980).
[5] M. Paulus and O. J. F. Martin, Phys. Rev. E **63**, 066615 (2001).
[6] K. M. Mitzner, J. Math. Phys. **5**, 1776 (1964).
[7] J. M. Wylie and J. E. Sipe, Phys. Rev. A **30**, 1185 (1984).
[8] R. R. Chance, A. Prock, and R. Silbey, Adv. Chem. Phys. **60**, 1 (1978).
[9] S. M. Barnett, B. Huttner, and R. Loudon, Phys. Rev. Lett. **68**, 3698 (1992).
[10] S. Barnett, B. Huttner, R. Loudon, and R. Matloob, J. Phys. B **29**, 3763 (1996).
[11] R. Sprik, B. van Tiggelen, and A. Lagendijk, Europhys. Lett. **35**, 265 (1996).
[12] A. Tip, Phys. Rev. A **57**, 4818 (1998).
[13] H. T. Dung, L. Knoll, and D.-G. Welsch, Phys. Rev. A **62**, 053804 (2000).
[14] A. Tip, L. Knöll, S. Scheel, and D.-G. Welsch, Phys. Rev. A **63**, 043806 (2001).
[15] A. Tip, Phys. Rev. A **69**, 013804 (2004).
[16] A. Tip, J. Math. Phys. **47**, 012902 (2006).
[17] M. Reed and B. Simon, *Fourier Analysis and Self-Adjointness* (Academic, New York, 1975).
[18] A. Tip, A. Moroz, and J. M. Combes, J. Phys. A **33**, 6223 (2000).
[19] M. Reed and B. Simon, *Functional Analysis* (Academic, New York, 1975).